4. Komplexní číslo sdružené, dělení komplexních čísel

- KOMPLEXNÍ ČÍSLO SDRUŽENÉ (KOMPLEXNĚ SDRUŽENÉ ČÍSLO) s číslem $z = a + bi$ nazýváme číslo $\bar{z} = a - bi$ (zn. $\bar{z}$ … z s pruhem) $a, b \in R$

(změníme znaménko u imaginární části – POZOR, neplést s opačným čís.)

př. $z_1 = 2 + 3i$ $\quad$ $\bar{z}_1 = 2 - 3i$ kompl. sdruž.

($z_1' = -2 - 3i$ … opačné)

[$\overline{2+3i} = 2 - 3i$]

$z_2 = 1 - i$ $\quad$ $\bar{z}_2 = 1 + i$

[$\overline{1-i} = 1 + i$ … jiný zápis]

$z_3 = \frac{3}{2}i$ $\quad$ $\bar{z}_3 = -\frac{3}{2}i$

$z_4 = -25$ $(= -25 + 0i)$ $\quad$ $\bar{z}_4 = -25$ $(-25 - 0i)$

př. $\overline{(\overline{2-3i})} = \overline{(2+3i)} = 2 - 3i$

$\overline{(\bar{z})} = z$

! POZOR NA SPRÁVNÉ ZÁPISY !

- pro $\forall\, z = a + bi \in C$:

$\overline{(\bar{z})} = z$

$z \cdot \bar{z} = (a + bi)(a - bi)$

$\downarrow$ $z \cdot \bar{z} = a^2 + \underbrace{b^2}_{\geq 0}$ $\rightarrow$ $z \cdot \bar{z} \geq 0$

$\downarrow$ $z \cdot \bar{z} \in R$

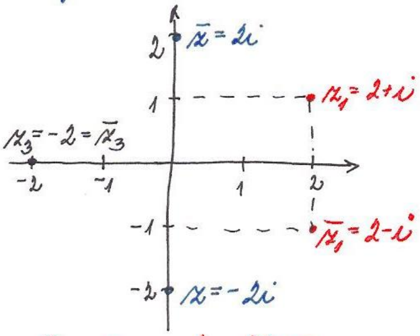
$z \cdot \bar{z} > 0$ pro $z \neq 0$

$z \cdot \bar{z} = 0$ pro $z = 0$

- pro $\forall\, z_1, z_2 \in C$:

$\overline{-z_1} = -\bar{z}_1$ $\qquad$ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ $\qquad$ $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$ $\qquad$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ pro $z_2 \neq 0$

- v Gaussově rovině



komplexně sdružená čísla jsou SOUMĚRNÁ PODLE REÁLNÉ OSY

(pozor! OPAČNÁ – podle počátku soust. souř.)

- PŘEVRÁCENÉ ČÍSLO naz. $z' = \frac{1}{z}$, kde $z \neq 0$

$$z' = \frac{1}{a+bi} = \frac{1}{\underbrace{a+bi}_{z}} \cdot \frac{a-bi}{\underbrace{a-bi}_{\bar{z}}} = \frac{a-bi}{a^2 - b^2 i^2} = \frac{a-bi}{a^2+b^2} = \underbrace{\frac{a}{a^2+b^2}}_{Re(z')} - \underbrace{\frac{b}{a^2+b^2}}_{Im(z')} i$$

$\swarrow$ musíme upravit na algebr. tvar $z' = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}}$

př. $z = 3 + 2i$

$$z' = \frac{1}{3+2i} = \frac{1}{3+2i} \cdot \frac{3-2i}{3-2i} = \left(\frac{3-2i}{3^2 - 2^2 i^2}\right) = \frac{3-2i}{9 - \underbrace{4i^2}_{+4}} = \frac{3-2i}{13} = \underbrace{\frac{3}{13}}_{Re} \underbrace{- \frac{2}{13}}_{Im} i$$

- PODÍL $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2}$ (součin z_1 a čísla převráceného k z_2)

! ANI V OBORU KOMPLEXNÍCH ČÍSEL NELZE DĚLIT NULOU !

př. $z = \frac{1-i}{3+i} = \frac{1-i}{3+i} \cdot \frac{3-i}{3-i} = \frac{3-i-3i+i^2}{9-i^2} = \frac{2-4i}{10} = \frac{2(1-2i)}{10} =$

($3+i$: $a+bi$; $3-i$: $a-bi$; $\frac{3-i}{3-i} = 1$; $i^2 = -1$)

$= \frac{1-2i}{5} = \frac{1}{5} - \frac{2}{5}i$ ($[\frac{1}{5}, -\frac{2}{5}]$ v Gaus. r.)

- pro úpravu na algebr. tvar využíváme $\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\overline{z_2}}{\overline{z_2}}$ ($z_2 \neq 0$)

- MOCNINA S CELÝM EXPONENTEM

- pro $n \in N$: jako s přirozeným exponentem
- dodefinujeme $z^0 = 1$ ($z \neq 0$)

$z^{-n} = \left(\frac{1}{z}\right)^n = \frac{1}{z^n}$ $n \in N \Rightarrow -n \in Z^-$ ($z \neq 0$)

př. (1. zp.) $(1-i)^{-2} = \frac{1}{(1-i)^2} = \frac{1}{1-2i+i^2} = \frac{1}{-2i} \cdot \frac{2i}{2i} = \frac{2i}{-4i^2} = \frac{2i}{4} = \frac{i}{2}$ ($[0, \frac{1}{2}]$)

(stačí i jen i)

(2. zp.) $(1-i)^{-2} = \left(\frac{1}{1-i}\right)^2 = \left(\frac{1}{1-i} \cdot \frac{1+i}{1+i}\right)^2 = \left(\frac{1+i}{1-i^2}\right)^2 = \left(\frac{1+i}{2}\right)^2 = \frac{1+2i+i^2}{4} =$

$= \frac{2i}{4} = \frac{i}{2}$

Příklady

① Urči podíl

3b/22 a) $\frac{3+4i}{2-5i} = \frac{3+4i}{2-5i} \cdot \frac{2+5i}{2+5i} = \frac{6+15i+8i+20i^2}{4+25} = \frac{-14+23i}{29}$

$= \frac{-14}{29} + \frac{23}{29}i$

b) $\frac{1}{2+i} = \frac{1}{2+i} \cdot \frac{2-i}{2-i} = \frac{2-i}{4-i^2} = \frac{2-i}{5} = \frac{2}{5} - \frac{i}{5} = \frac{2}{5} - \frac{1}{5}i$

c) $\frac{1-i}{i} = \frac{1-i}{i} \cdot \frac{-i}{-i} = \frac{-i+i^2}{-i^2} = \frac{-i-1}{1} = -1-i$

MS4 3.27 d) $1 - \frac{1}{i+1} + (i-1)i = 1 - \frac{1}{1+i} \cdot \frac{1-i}{1-i} + i^2 - i = -\frac{1-i}{1-i^2} - i =$

$= -\frac{1-i}{2} - i = \frac{-1+i}{2} - i = \frac{-1+i-2i}{2} = \frac{-1-i}{2} = -\frac{1}{2} - \frac{i}{2} = -\frac{1}{2} - \frac{1}{2}i$

c) $\frac{i+1}{i} : \frac{i-1}{2i+3} = \frac{i+1}{i} \cdot \frac{2i+3}{i-1} = \frac{2i^2+3i+2i+3}{i^2-i} = \frac{1+5i}{-1-i} \cdot \frac{-1+i}{-1+i} =$

$= \frac{-1+i-5i+5i^2}{(-1)^2-i^2} = \frac{-6-4i}{1+1} = \frac{-6-4i}{2} = \frac{2(-3-2i)}{2} = -3-2i$

(2. zp.)
$\frac{i+1}{i} : \frac{i-1}{2i+3} = \left(\frac{i+1}{i} \cdot \frac{-i}{-i}\right) \cdot \left(\frac{2i+3}{-1+i} \cdot \frac{-1-i}{-1-i}\right) = \frac{-i^2-i}{-i^2} \cdot \frac{-2i-2i^2-3-3i}{1-i^2} =$

$= (1-i) \cdot \frac{-1-5i}{2} = \frac{(1-i)(-1-5i)}{2} = \frac{-1-5i+i+5i^2}{2} = \frac{-6-4i}{2} = -3-2i$

[úpravy lze provádět různě]

② Vypočtěte

4P/23 a) $(1+i)^{-2} = \left(\frac{1}{1+i}\right)^2 = \frac{1}{(1+i)^2} = \frac{1}{1+2i+i^2} = \frac{1}{2i} \cdot \frac{(-i)}{(-i)} = \frac{-i}{-2i^2} = \frac{-i}{2} = -\frac{1}{2}i$

[nebo $\frac{-2i}{-2i}$]

b) $\left(\frac{1+i}{1-i}\right)^{-1} = \left(\frac{1-i}{1+i}\right)^1 = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-2i+i^2}{1-i^2} = \frac{-2i}{2} = -i$ $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

nebo $= \left(\frac{1+i}{1-i} \cdot \frac{1+i}{1+i}\right)^{-1} = \left(\frac{1+2i+i^2}{1-i^2}\right)^{-1} = \left(\frac{2i}{2}\right)^{-1} = i^{-1} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{-i^2} = -i$

③ Zapište v algebraickém tvaru

1/46 DÚ a) $\frac{2-i}{-3+i} - \frac{1+2i}{1-3i} = \frac{2-i}{-3+i} \cdot \frac{-3-i}{-3-i} - \frac{1+2i}{1-3i} \cdot \frac{1+3i}{1+3i} = \frac{-6-2i+3i+i^2}{9-i^2} -$

$- \frac{1+3i+2i+6i^2}{1-i^2} = \frac{-7+i}{10} - \frac{(-5+5i)}{10} = \frac{-7+5-4i}{10} = \frac{-2-4i}{10} = \frac{-1-2i}{5}$

$= -\frac{1}{5} - \frac{2}{5}i$ [nebo pracovat jako s dvojčleny - na společ. jmen. => na alg. tvar]

b) $2 + \frac{1}{3i} - \frac{3}{i^3} = 2 + \frac{1}{3i} - \frac{3}{-i} = 2 + \frac{1}{3i} + \frac{3}{i} = \frac{6i+1+9}{3i} = \frac{10+6i}{3i} \cdot \frac{-3i}{-3i} =$

[$i^3 = -i$] [stačí $i \frac{-i}{-i}$]

$= \frac{-30i-18i^2}{9} = \frac{18-30i}{9} = 2 - \frac{10}{3}i$

c) $\frac{10}{3-2i} + \frac{3-2i}{10} = \frac{10(3+2i)}{(3-2i)(3+2i)} + \frac{3-2i}{10} = \frac{30+20i}{9-4i^2} + \frac{3-2i}{10} =$

$= \frac{30+20i}{13} + \frac{3-2i}{10} = \frac{10(30+20i)+13(3-2i)}{130} = \frac{300+200i+39-26i}{130} =$

$= \frac{339+174i}{130} = \frac{339}{130} + \frac{174}{130}i$

[lze i nejprve zlomky sečíst => na alg. tvar]

d) $\left(\frac{1-i}{1+i}\right)^{-2}+\left(\frac{1-i}{1+i}\right)^{2}=\left(\frac{1+i}{1-i}\right)^{2}+\left(\frac{1-i}{1+i}\right)^{2}=\frac{1+2i+\overset{-1}{i^2}}{1-2i+\underset{-1}{i^2}}+\frac{1-2i+\overset{-1}{i^2}}{1+2i+\underset{-1}{i^2}}=$

$=\frac{2i}{-2i}+\frac{-2i}{2i}=-1-1=\underline{-2}$

④ V algebraickém tvaru vyjádřete čísla sdružená s čísly

1.18 a) $z=(2+i\sqrt{3})(3-i\sqrt{2})-i=6-2\sqrt{2}i+3\sqrt{3}i-\underbrace{i^2\sqrt{6}}_{+\sqrt{6}}-i=6+\sqrt{6}+$
upravíme nejprve z na alg. tvar => $\bar{z}$
$+i(3\sqrt{3}-2\sqrt{2}-1)$

$\underline{\bar{z}=6+\sqrt{6}-(3\sqrt{3}-2\sqrt{2}-1)i}$

b) $z=\frac{2i}{1-i}=\frac{2i(1+i)}{(1-i)(1+i)}=\frac{2i+2\overset{-2}{i^2}}{1-\underbrace{i^2}_{+1}}=\frac{2i-2}{2}=\frac{2(i-1)}{2}=i-1=-1+i$

$\underline{\bar{z}=-1-i}$

c) $z=(1-i\sqrt{3})^2=1-2i\sqrt{3}+(i\sqrt{3})^2=1-2i\sqrt{3}+\underbrace{i^2 3}_{-3}=-2-2\sqrt{3}i$

$\underline{\bar{z}=-2+3\sqrt{3}i}$

d) $\pi-2+i+\frac{1}{i}=\pi-2+i+\frac{1\cdot(-i)}{\underbrace{i\cdot(-i)}_{+1}}=-2+\pi+i-i=\pi-2\ (\in R)$

$\underline{\bar{z}=\pi-2}$ $\qquad (z=\underbrace{\pi-2}_{Re}+0i \quad Im=0)$

⑤ Řešte v C

a) $(2+3i)z+iz=1-i$ $\qquad O=C,\ D=C$

B 13.1 ř c)

$2z+3zi+iz=1-i$

$2z+4zi=1-i$

$z(2+4i)=1-i$

$z=\frac{1-i}{2+4i}=\frac{1-i}{2+4i}\cdot\frac{2-4i}{2-4i}=$

$=\frac{2-4i-2i+4\overset{-4}{i^2}}{4-\underbrace{16i^2}_{+16}}=\frac{-2-6i}{20}$

$=\frac{2(-1-3i)}{20}=\frac{-1-3i}{10}$

$z=-\frac{1}{10}-\frac{3}{10}i$

$\underline{K=\{-\frac{1}{10}-\frac{3}{10}i\}}$

[$z=\frac{1-i}{2+4i}=\frac{1-i}{2(1+2i)}\cdot\frac{1-2i}{1-2i}$
menší čísla]

b) $\frac{i}{1-2i}x+2i=x-\frac{1}{2+i}$ $\qquad O=C,\ D=C$

$ix(2+i)+2i(1-2i)(2+i)=x(1-2i)(2+i)-1(1-2i)$

$2xi+i^2x+2i(\underbrace{2+i-4i-2i^2}_{4-3i})=x(\underbrace{2+i-4i-2i^2}_{4-3i})-1+2i$

$2xi-x+8i-6i^2=4x-3xi-1+2i$

$2xi-x-4x+3xi=-1-2i-8i-6$

$5xi-5x=-7-6i$

$x(5i-5)=-7-6i$

$x=\frac{-7-6i}{-5+5i}$ [nebo $\frac{7+6i}{5-5i}$ => „lepší" znaménka]

$x=\frac{-7-6i}{-5+5i}\cdot\frac{-5-5i}{-5-5i}=$

$=\frac{35+35i+30i+30i^2}{25+25}=\frac{5+65i}{50}$

$x=\frac{1+13i}{10}=\frac{1}{10}+\frac{13}{10}i$

$\underline{K=\{\frac{1}{10}+\frac{13}{10}i\}}$

⑥ Řešte v $\mathbb{C}$

1.19/25 a) $\frac{1+2i}{1-i}x + \frac{1}{1+i} = 0$ $\qquad O=\mathbb{C},\ D=\mathbb{C}$

$(1+2i)(1+i)x + (1-i)1 = 0$

$(1+i+2i+2i^2)x + 1 - i = 0$ $\quad (2i^2 = -2)$

$(-1+3i)x + 1 - i = 0$

členy s nezn. na jednu stranu, bez nezn. na druhou

$(-1+3i)x = -1+i$

$x = \frac{-1+i}{-1+3i}$

převést na alg. tvar

$\left[x = \frac{-1+i}{-1+3i} \cdot \frac{-1-3i}{-1-3i} = \dots \right]$

lépe

$x = \frac{-1+i}{-1+3i} = \frac{-1(1-i)}{-1(1-3i)} = \frac{(1-i)}{1-3i} \cdot \frac{(1+3i)}{1+3i} =$

$= \frac{1+3i-i-3i^2}{1-9i^2} = \frac{4+2i}{10} = \frac{2(2+i)}{10}$ $\quad (-3i^2 = +3,\ -9i^2 = +9)$

$x = \frac{2+i}{5}$ $\qquad K = \{\frac{2}{5} + \frac{1}{5}i\}$

$x = \frac{2}{5} + \frac{i}{5}$

b) $\frac{2i}{1+i}x - \frac{3-i}{2-3i} = i$ $\qquad O=\mathbb{C},\ D=\mathbb{C}$

$2ix(2-3i) - (3-i)(1+i) = i(1+i)(2-3i)$

$4xi - 6i^2x - (3+3i-i-i^2) = i(2-3i+2i-3i^2)$ $\quad (-6i^2x = +6x,\ -3i^2 = +3)$

$4ix + 6x - 4 - 2i = 5i - i^2$ $\quad (-i^2 = +1)$

$x(6+4i) = 5i+1+4+2i$

$x = \frac{5+7i}{6+4i}$

$x = \frac{5+7i}{6+4i} \cdot \frac{6-4i}{6-4i} = \frac{30-20i+42i-28i^2}{36-16i^2}$ $\quad (-28i^2 = +28,\ -16i^2 = +16)$

$x = \frac{58+22i}{52} = \frac{29}{26} + \frac{11}{26}i$

$\left[\text{lze i } x = \frac{5+7i}{6+4i} = \frac{5+7i}{2(3+2i)} \cdot \frac{(3-2i)}{(3-2i)} = \dots \right]$

$K = \{\frac{29}{26} + \frac{11}{26}i\}$

c) $\frac{x}{1+i\sqrt{2}} = ix + 1$

$x = (ix+1)(1+i\sqrt{2})$

$x = ix + i^2\sqrt{2}x + 1 + i\sqrt{2}$ $\quad (i^2 = -1)$

$x = ix - \sqrt{2}x + 1 + i\sqrt{2}$

$x + \sqrt{2}x - ix = 1 + i\sqrt{2}$

$x(1+\sqrt{2}-i) = 1 + i\sqrt{2}$

$x = \frac{1+i\sqrt{2}}{1+\sqrt{2}-i}$

$x = \frac{1+i\sqrt{2}}{1+\sqrt{2}-i} \cdot \frac{1+\sqrt{2}+i}{1+\sqrt{2}+i} =$

$= \frac{1+\sqrt{2}+i+i\sqrt{2}+i\sqrt{2}\sqrt{2}+i^2\sqrt{2}}{1+\sqrt{2}+i+\sqrt{2}+\sqrt{2}\cdot\sqrt{2}+i\sqrt{2}-i-i\sqrt{2}-i^2}$ $\quad (i\sqrt{2}\sqrt{2} = 2i,\ i^2\sqrt{2} = -\sqrt{2},\ \sqrt{2}\cdot\sqrt{2} = 2,\ -i^2 = +1)$

$= \frac{1+3i+\sqrt{2}i}{4+2\sqrt{2}} = \frac{1+(3+\sqrt{2})i}{4+2\sqrt{2}}$

$x = \frac{1}{4+2\sqrt{2}} + \frac{3+\sqrt{2}}{4+2\sqrt{2}}i$ (lze dále usměrnit - odstr. odmocniny ze jmenovatele)

$K = \{\frac{1}{4+2\sqrt{2}} + \frac{3+\sqrt{2}}{4+2\sqrt{2}}i\}$

d) $x(\sqrt{2}+i)^2 = (\frac{1}{i}+2)^2$

$x(2+2\sqrt{2}i+i^2) = \left(\frac{1+2i}{i} \cdot \frac{-i}{-i}\right)^2$ $\quad (i^2 = -1,\ -i^2 = 1)$

$x(1+2\sqrt{2}i) = (-i-2i^2)^2$ $\quad (-2i^2 = +2)$

$x(1+2\sqrt{2}i) = (2-i)^2$

$x(1+2\sqrt{2}i) = 4-4i+i^2$ $\quad (i^2 = -1)$

$x(1+2\sqrt{2}i) = 3-4i$

lze upravovat různě:

$(\frac{1}{i}+2)^2 = (\frac{1}{i} \cdot \frac{-i}{-i} + 2)^2 = (2-i)^2$

$x = \frac{3-4i}{1+2\sqrt{2}i}$

$x = \frac{3-4i}{1+2\sqrt{2}i} \cdot \frac{1-2\sqrt{2}i}{1-2\sqrt{2}i} =$

$= \frac{3-6\sqrt{2}i-4i+8\sqrt{2}i^2}{1-4\sqrt{2}\sqrt{2}i^2} =$ $\quad (i^2 = -1,\ \sqrt{2}\sqrt{2} = 2,\ -4\sqrt{2}\sqrt{2}i^2 = +8)$

$= \frac{3-8\sqrt{2}-i(6\sqrt{2}+4)}{9}$

$x = \frac{3-8\sqrt{2}}{9} - \frac{6\sqrt{2}+4}{9}i$

$K = \{\frac{3-8\sqrt{2}}{9} - \frac{6\sqrt{2}+4}{9}i\}$

7) Řešte v C

a) $(5-\frac{1}{i})\bar{z}+2z=22i$ $\quad O=\mathbb{C}$, $D=\mathbb{C}$

pokud v rci $z, \bar{z}$
=> nahradit
$z = a+bi$
$\bar{z} = a-bi$
⇓
rovnost komplex. čísel
($a,b \in R$ neznámé)
=> $z=a+bi$

(*) $\frac{5i-1}{i}(a-bi)+2(a+bi)=22i$

$\frac{5i-1}{i}\cdot\frac{-i}{-i}(a-bi)+2(a+bi)=22i$

$\left(\frac{\overset{5}{-5i^2}+i}{\underset{1}{-i^2}}\right)(a-bi)+2(a+bi)=22i$

$(5+i)(a-bi)+2(a+bi)=22i$

$5a-5bi+ai-\underbrace{bi^2}_{+b}+2a+2bi=22i$

$\underbrace{7a+b}_{Re}+(\underbrace{a-3b}_{Im})i=22i \quad (\underset{Re}{0}+\underset{Im}{22i})$

rovnost kompl. čísel

$7a+b=0 \quad /\cdot 3$
$a-3b=22$

$21a+3b=0$
$a-3b=22$

$22a=22$
$a=1$

$b=-7a=-7$

$z=1-7i$ $\qquad K=\{1-7i\}$

> (*) $(5-\frac{1}{i})\bar{z}+2z=22i$ ASI LEPŠÍ ZPŮSOB
>
> $(5-\frac{1}{i}\cdot\frac{-i}{-i})(a-bi)+2(a+bi)=22i$
>
> $(5-\frac{-i}{\underset{1}{-i^2}})(a-b)+2(a+bi)=22i$
>
> $(5+i)(a-b)+2(a+bi)=22i$

b) $(5-\frac{1}{i})\bar{z}=z(1-i)+12$ $\quad O=\mathbb{C}$, $D=\mathbb{C}$

$(5-\frac{1}{i}\cdot\frac{-i}{-i})(a-bi)=(a+bi)(1-i)+12$

$(5-\frac{-i}{\underset{1}{-i^2}})(a-bi)=(a+bi)(1-i)+12$

$(5-(-i))(a-bi)=(a+bi)(1-i)+12$

$(5+i)(a-bi)=(a+bi)(1-i)+12$

$5a-5bi+ia-\underbrace{bi^2}_{+b}=a-ai+bi-\underbrace{bi^2}_{+b}+12$

$5a-a+2ai-5bi-bi-12=0$

lze taky vše převést na jednu stranu a pak porovnat Re, Im

$4a+(2a-6b)i-12=0$

$\underbrace{4a-12}_{Re}+(\underbrace{2a-6b}_{Im})i=0 \quad \underset{Re}{0}+\underset{Im}{0i}$

rovnost kompl. čísel

$4a-12=0$
$2a-6b=0$

$4a=12$
$a=3$

$2a-6b=0$
$6b=2a$
$6b=2\cdot 3$
$6b=1$
$b=1$

$z=a+bi$
$z=3+i$ $\qquad K=\{3+i\}$